Yiddish word of the day

"fregn a frageh" = פֿרעגן אַ פֿראַגע

to ask a question =

Yiddish phrase of the day

Az me fregt, blonzhet men nisht = אַז מע פֿרעגט, בלאָנדזשעט מען נישט!

"As one asks, one cannot err" =

Recall from last time

1) Given linear system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned} \longleftrightarrow A\vec{x} = \vec{b}$$

2) We saw we could mult matrices, but they "behave" diff than mult of #'s.

- We had two non-zero matrices A, B such that $AB = 0$ $\longleftarrow$ the "0 matrix"

- We had 3 matrices A, B, C with $B \neq C$ and $AB = AC$ (could not "cancel" the A)

1) Note the similarities of $A\vec{x} = \vec{b}$ with $ax = b$

- note $ax = b$ has a unique solution precisely when $a \neq 0$ $\left(x = \frac{1}{a} \cdot b\right)$

Q: Can we "divide" by matrix A?

Def: Let A $n \times n$ matrix. Then A is invertible if there exists some other matrix, denoted A^{-1} such that

$$A^{-1}A = AA^{-1} = \mathbb{1}_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n \times n}$$

identity matrix

- Note: $\mathbb{1}_n \vec{x} = \vec{x}$ and $\mathbb{1}_n B = B$ Check!

Note: If matrix A is invertible then the linear system $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$

$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

Some consequences of being an invertible matrix

Have 2 old results. Let A $n \times n$ matrix.

1) the columns of A span $\mathbb{R}^n$ iff the matrix eqn $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$.

2) the columns of A are LI iff
$\text{null}(A) = \{\vec{0}\}$
(also equivalently, whenever $A\vec{x} = \vec{b}$ has a solution, it will be unique)

Thus, since we saw $A\vec{x} = \vec{b}$ has unique solution for any $\vec{b}$ if A is invertible, then,
the columns of A are a basis !!!

- Invertible matrix A $=$ a matrix whose columns are a basis.

A couple of results

1) If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$

2) If A, B are 2 invertible $n \times n$ matrices, then AB is invertible too, with $(AB)^{-1} = B^{-1}A^{-1}$

- $(AB)(B^{-1}A^{-1}) = A(I_n)A^{-1} = AA^{-1} = I_n$

- $(B^{-1}A^{-1})(AB) = B^{-1} I_n B = B^{-1}B = I_n$

3) Suppose A is invertible Then multiplication behaves more like how we are familiar.

ex) Suppose A invertible and we have 2 matrices B, C with $AB = AC$

then ..... $B = C$

ex) Similar statement about "zero divisors"

See HW

Q2: How to determine if matrix is invertible, and how to find the inverse if it is.

A) First in case A is 2×2 matrix.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then A is invertible when

$ad - bc \neq 0$

and $\boxed{A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

ex) $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$. Find A^{-1} if it exists.

$(1)(3) - 2(-1) = 3 + 2 = 5$

$A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{pmatrix}$

<u>Check</u>: $A^{-1}A = \begin{pmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$AA^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Now more generally

Let A be an $n \times n$ matrix

1) Augment A by $\mathbb{1}_n$

$$\left(\begin{array}{c|ccc} A & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{array}\right)$$

2) Put A into Reduced Row Echelon Form

- if at any point, you get a 0 row on left

  STOP, A is not-invertible !!

- else finish putting it into reduced row ech form

$$\left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & B \\ 0 & & 1 & \end{array}\right)$$

This matrix B is A^{-1} !!

ex) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 2 \end{pmatrix}$ Find A^{-1} if it exists.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 4 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -2 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3/2]{R_2 \to R_2/2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1 & -1/2 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1 & -1/2 \end{array}\right)$$

Claim $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & -1/2 & 1/2 \\ 1/2 & 1 & -1/2 \end{pmatrix}$ — Skip the checking

ex) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$ Find A^{-1} if it exists.

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right)$

After some work

get a zero row, not invertible.

- A sneaky way: col(3) = 2 col(2) so columns not LI

Elementrary Matrices

- These matrices will do the following
  - Justify this invertible matrix algorithm.
  - Justify our row operations
    - lead to some new ideas (see HW)
  - Provide a bridge to determinants

Recall the 3 elementary row operations

1) Scaling a row $\longleftrightarrow$ $D_i^n(c)$
2) Swap a row $\longleftrightarrow$ P_{ij}^n
3) Add multiples of 1 row to another $\longleftrightarrow$ $T_{ij}^n(c)$

Associated to these 3 operations are the following 3 matrices

1) $D_i^n(c)$ = the $n \times n$ identity matrix, except in row i where it is c and not 1

ex) $D_2^3(1/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$D_3^4(-12) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$D_2^5(5) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2) $P_{i,j}^n$ = $n \times n$ identity matrix but swap row i and row j

ex) $P_{12}^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

ex) $P^5_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

ex) $P^4_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

3) $T^n_{ij}(c)$ = $n \times n$ identity but add $c \times$ row i to row j

ex) $T^2_{12}(2) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$$T_{23}^{3}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$T_{24}^{4}(-6) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -6 & 0 & 1 \end{pmatrix}$$

$$T_{35}^{5}(4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 1 \end{pmatrix}$$

Q - Who the hell cares?

ex) $\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{= P^2_{12}} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

ex) $\underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}_{= D^2_1(3)} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 9 \\ 4 & 1 \end{pmatrix}$

ex) $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & -5 \end{pmatrix}$

$\quad \shortparallel$

$T_{12}^{2}(-2)$

Q: What does $R_2 \to R_2 - 2R_1$ give?

Doing row operations to our matrix

$\shortparallel$

Multiplying my matrix ON THE LEFT by an Elementary Matrix !!

Note: • Doing row operations to a matrix is a reversable process

• So thinking of row operations as matrix mult, the fact that they are reversable means

the elementary matrices are invertible.

1) $\left(D_i^n(c)\right)^{-1} = D_i^n\left(\frac{1}{c}\right)$

ex) $D_i^2(2) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$$(D_1^4(2))^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \quad \underline{\text{Check!}}$$

2) $(P_{ij}^n)^{-1} = P_{ji}^n = P_{ij}^n$; Ex) Check that $(P_{12}^3)(P_{12}^3) = I_{3\times 3}$

3) $(T_{ij}^n(c))^{-1} = T_{ij}^n(-c)$

ex) Check $T_{12}^3(2)T_{12}^3(-2) = I_{3\times 3}$

$T_{12}^3(-2)T_{12}^3(2) = I_{3\times 3}$